Hinrichs (Gus,)

THE METHOD

OF

QUANTITATIVE INDUCTION

IN

PHYSICAL SCIENCE.

A GUIDE FOR STUDENTS IN THE LABORATORY
AND LECTURE ROOM.

BY

DR. GUSTAVUS HINRICHS.

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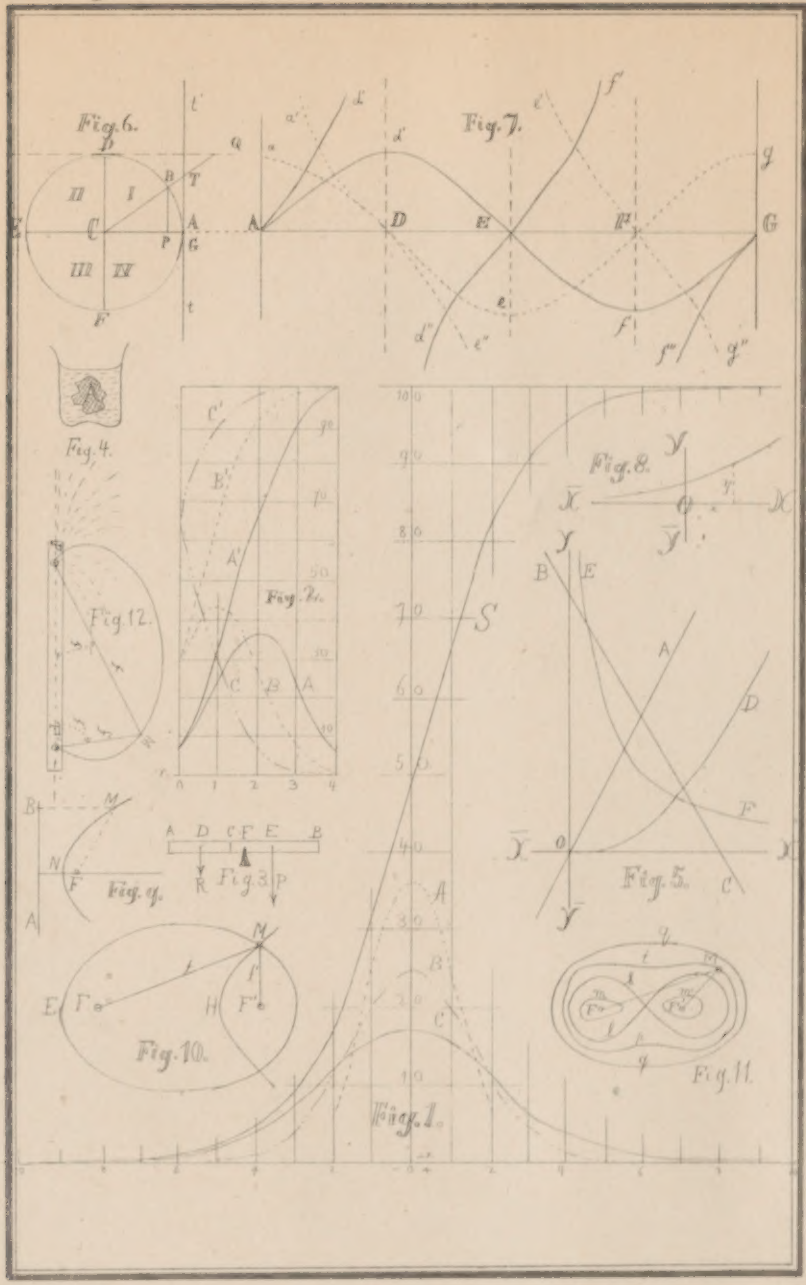
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
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Fig. 6.
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Fig. 8.
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Fig. 10.
Fig. 11.
Fig. 12.

HINRICHS' METHOD

OF

QUANTITATIVE INDUCTION

IN

PHYSICAL SCIENCE.

It is a most certain and exact thing.

KEPLER, Harm. Mund., 1619.

THE PRINCIPLES

OF

PHYSICAL SCIENCE,

DEMONSTRATED BY

THE STUDENT'S OWN EXPERIMENTS AND OBSERVATIONS.

BY

DR. GUSTAVUS HINRICHS.

IN THREE VOLUMES.

VOL. I., PART I.

THE METHOD OF

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AND LECTURE ROOM.

BY

DR. GUSTAVUS HINRICHS,
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Koenigsberg, Emden, Cherbourg, etc.

WITH ONE PLATE, JOURNAL OF EXPERIMENTS,
AND METRICAL CO-ORDINATE SHEETS.

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I. INTRODUCTION.

1. In the Principles of Physics the methods of physical research shall be described and exemplified for students' use in the physical laboratory, and a synopsis of the most important general laws of physics shall be given as a guide to Lectures. Compare El. Phys. 451–561.

This part first of the Principles of Physics will mainly treat of the method of induction.

2. In physical research two consecutive stages may be distinguished, the qualitative and the quantitative stage. First, the phenomena are distinguished from one another merely in regard to quality or kind; thereafter, by a more careful investigation, these qualitative differences are found to be expressible in numbers, or in quantity. Numerous examples of these two stages of investigation occur already in the Elements of Physics. Thus the pressure of the air is qual. in 127, quant. in 128; pitch of tones, qual. in 141, quant. in 142; optical image, qual. in 241, quant. in 242; solubility, qual. in 151, 152, and quant. in 156. The same two stages are marked in the Elements of Chemistry (qual. 32, quant. 33; qual. 42, quant. 43–51; etc., etc). Find additional examples.

3. The quantitative stage is always the more difficult and therefore is preceded by the qualitative stage in the historic development of Physical Science. Thus qualitative chemical analysis precedes quantitative chemical analysis. In the elements of physics only the qualitative stage of the laws of magnetism and electricity has been given (in Chapter V), the quantitative stage being

deferred to the principles of Physics and to special treatises on electricity and magnetism.

4. Two general methods of research are used, the inductive and deductive methods.

INDUCTION ascends from particular phenomena to general laws or principles; from observed and measured special quantities to the general mathematical relations between them, constituting the mathematical expression of the laws. Laws thus found are of course proved by the experiments and observations from which they were obtained. See El. Phys. 41–42, 43, 44, 48.

DEDUCTION descends from some general principle assumed as hypothesis or as axiom (El. Phys. 87 and 89), to the general relations and special phenomena which follow from the general principle. Such deduction to be valid must be strictly exact, therefore mathematical, either in algebraical symbols (analytical deduction—see El. Phys. 90; 92–93; 94), or in geometrical constructions (geometrical deductions—see El. Phys. 257). But these deductions only become properly parts of physics, when they have been verified by experiment or observation (see experiments in El. Phys. 90–91; 92–93; 94–95 for the three mechanical powers). If the phenomena observed, both qualitatively and quantitatively, correspond to the mathematical deductions made, then these latter are accepted as physical deductions, and the hypothesis from which they are derived becomes a principle of physics.

5. A great system of such verified deductions derived from one single hypothesis constitutes a physical theory. Examples: Universal gravitation; mechanical theory of heat; undulatory theory of light; dynamical theory of electricity. Compare El. Phys. page 158, note.

6. Galileo Galilei (1564–1642) first practically and successfully used both these methods in his investi-

gations of the laws of falling bodies, the motions of projectiles, the laws of the pendulum, and the resistance of materials. His great work, containing these investigations, was published 1638, in Leyden, Holland, under the title Discorsi e Demostrationi matematiche intorno a due nuove Scienze, attenenti alla mecanica, ei movimenti locali. This work is therefore the corner-stone of modern physics, and Galileo the father of modern physical science. The motto of our Elements of Physics has been taken from this work of Galileo (page 132, of Bologna, edition 1665).

Lord Bacon, of Verulam (1561–1626), is by some English writers praised as the founder of modern science. But he was simply a verbose speculative metaphysician, ignorant of the very rudiments of quantitative physical research. He could not even understand the simple laws of perspective involved in the Copernican system of planet-motions. Modern science owes not the slightest result to him, and would therefore not be lacking of its present state of advancement if the courtier-metaphysician, Lord Bacon, never had existed.*

* An examination of any of Bacon's writings by a person who knows the least of physical science, will substantiate the above. For those students who may be unwilling to waste so much time, we reprint the following from Bacon's (so called) Natural History, wherein he records one thousand experiments, divided into ten sections of one hundred each, called Centuries. All comment on such "experiments" would be superfluous.

EXPERIMENT SOLITARY TOUCHING WEIGHT.

789. Weigh iron and aqua fortis severally; then dissolve the iron in the aqua fortis, and weigh the dissolution; and you shall find it to bear as good weight as the bodies did severally; notwithstanding a good deal of waste by a thick vapour that issueth during the working; which showeth that the opening of a body doth increase the weight. This was tried once or twice, but I know not whether there were any error in the trial.

EXPERIMENTS IN CONSORT TOUCHING THE SECRET VIRTUE OF SYMPATHY AND ANTIPATHY.

997. The sympathy of individuals that have been entire, or have touched, is of all others the most incredible; yet according to our faithful manner of examination of nature, we will make some little mention of it. The taking away of warts,

7. To the above mentioned examples of deductive reasoning we shall here add Archimedes' demonstration of two simple laws of statics.*

I. The law of the Lever: The momentum of the power is equal to the momentum of the resistance. P.p=R.r (El. Phys. 95).

Hypothesis: A uniform prism has its center of gravity (El. Phys. 95), in the center of the form. Thus AB, Fig. 3, has its center of gravity in F and hence will be balanced about F in any position.

Deduction. Let the weight of each unit of length of this bar be w and the length of the bar 2 l, then FB=FA=l.

Suppose the bar to be divided at C and let CB=2b, and CA=2 a; consequently l=a+b. Since now the mere supposition of a division of the bar cannot disturb the position of the bar, the latter is still in equilibrium.

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by rubbing them with somewhat that afterwards is put to waste and consumed, is a common experiment; and I do apprehend the rather because of my own experience. I had from childhood a wart upon one of my fingers; afterwards, when I was about sixteen years old, being then at Paris, there grew upon both my hands a number of warts, at least one hundred in a month's space. The English ambassador's lady, who was a woman far from superstition, told me one day, she would help me away with my warts; whereupon she got a piece of lard with the skin on, and rubbed the warts all over with the fat side; and amongst the rest, that wart I had had from my childhood; then she nailed the piece of lard, with the fat side to the sun, upon a post of her chamber window, which was to the south. The success was, that within five weeks' space all the warts went quite away; and that wart which I had so long endured for company. But at the rest I did little marvel, because they came in so short a time, and might go away in a short time again; but the going away of that which had stayed so long doth yet stick with me. They say the like is done by the rubbing of warts with a green elder stick, and then burying the stick to rot in the muck. It would be tried with corns and wens, and such other excrescences. I would have it also tried with some parts of living creatures that are nearest to the nature of excrescences; as the combs of cocks, the spurs of cocks, the horns of beasts, &c. And I would try it both ways; both by rubbing those parts with lard, or elder, as before, and by cutting off some piece of those parts, and laying it to consume: to see whether it will work any effect towards the consumption of that part which was once joined to it.

*Archimedes lived in Syracuse, Sicily, B. C. 287-212; he is the founder of statics.

Now the weight of AC or $2wa=R$ may be be supposed applied in the center of gravity D of AC, which, according to the axiom, must be exactly midways between A and C, or $AD=DC=a$. That is, the leverarm of R is $r=FD=AF-AD=l-a=b$. Hence the momentum of R is $R.r=2w.a.\ b$. In the same manner the weight of CB is $2w.b=P$; the leverarm $p=FE =FB-BE=l-b=a$. Hence, for the lever AB in equilibrium, the momentum of P is $Pp=2wb.\ a$. Consequently $Pp=Rr$, which was to be demonstrated. Verify the same as a physical fact, see El. Phys. 95.

II. Archimedes' Principle (El. Phys. 123; state the same).

The liquid in any vessel (Fig. 4) be at rest; then the particles of the liquid do not move. We may now suppose, that the particles of any portion A of this liquid cannot move; this will evidently not change the actual equilibrium (axiom). But then this portion A is a solid of the density of the liquid, and exactly balanced by the surrounding liquid, neither moving up nor down. But the surrounding liquid will necessarily exert the same force on any other solid, exactly filling the space A; that is, such solid will be buoyed up by a force equal to the weight of the liquid displaced. This is the principle which was to be demonstrated. The verification of the same as physical fact, see El. Phys. 123.

8. The different branches of Physics are necessarily in very different stages of advancement. But the progress is always from qualitative to quantitative induction. Only in the more fully developed branches can deduction take place.

The progress of the student of physics should be guided by the teacher in a like manner; while the student himself should ascertain to which of these stages the representation of a given subject belongs.

II. INDUCTION.

9. The quantative inductive investigation of any series of phenomena can only be commenced after a qualitative research has shown that the variation of a certain quantity x determines the variation of some other quantity y. Then x is called the independent variable, y the dependent variable (El. Phys. 42). The same is expressed by saying y is a certain function of x and some constant c; which is written

$$y = c.f(x) \qquad (1)$$

The problem of quantitative induction is to find this function; or in words, to find in what manner y varies as function of x and what is the value of constant c.

10. For this purpose observe the numerical value of y corresponding to any given numerical value of x by making an experiment wherein x is made to have that precise value. In order to attain the greatest possible accuracy, repeat this experiment a number r times with the same value of x; the mean value of y obtained by these r experiments put down in your book as the value of y corresponding to the value of x used. Now give x some other value; find y by again repeating the experiment r times, and taking the mean of the values of y observed. Proceed in this manner, making usually not less than 10 such determinations; so that if each determination be the mean of 10 single experiments, you would have made 100 experiments.

The precise number r of such experiments to be made, will of course depend upon the phenomena investigated; no general rule can be given. If three distinct experiments give the same result y, of course only two or three experiments need to be made for each value of x.

The results thus obtained should be entered in a table of vertical columns; in the first column the

number (order) of the determination (No.); in the second the number r of experiments made for each one determination; in the third column the value x taken, and in the fourth the mean value of y obtained.

Remarks may be added in regard to various additional points, by using the number of the order (I column) for reference.

11. The value of x and y thus obtained should next be entered graphically as co-ordinates* according to a convenient scale (see El. Phys. 71 to 74), which scale MUST be stated on the chart. The scale may be—and usually has to be—different for the two variables. Thus, if x be the leverarm, and y the weight, then x may be $\frac{1}{2}$ (1 cm. leverarm to 0.5 cm. in drawing) and the scale of y may be 20 grams to 1 cm. Furthermore it is by no means necessary that the drawing should embrace the zero point of each or any of the co-ordinates; x might on the drawing begin with 10 cm., and y with 100 gr. This shows, however, that the principal lines (at least the red ones on the blanks given) should be marked by a number indicating the value which these lines represent.

Each determination should now be indicated by a point in the proper manner (El. Phys. 54), or by a faint cross passing through the point; and the order-number of the determination should be written near the same. This work must, of course, be done accurately and elegantly.

Thus an accurate graphical representation of all the experiments made will be obtained.

12. If now a sufficient number of points have been determined in the chart—as many as determinations made—then a continuous curve may be drawn so as to pass as near as possible to all the points. If the

* Square ruled paper, very convenient for this purpose, is added to this book.

different experiments were not affected by any errors, then such a curve would pass exactly through all these points. This curve we shall call the empirical curve. It is the geometrical representation of the analytical relation existing between the variables x and y observed. It now remains to find this analytical relation, in order to be able to express the same in words and to make use of the same for calculations in other cases like the given one.

13. All analytical relations or functions (see 9) are usually classified in three great groups, algebraic, transcendental, and trigonometrical functions. Here we shall consider only the first of these, because they are the simplest; the other two will be treated of in an appendix.

If the function is a simple algebraic one, then the empirical curve (Fig. 5) must be rectilinear (OA or BC), parabolic (OD), or hyperbolic (EF). These curves are characterized in the following articles.

14. The co-ordinates of a straight line change proportional to one another; that is (in BC)

$$\frac{y-b}{x-a}=c \tag{2}$$

or

$$y=q+cx \tag{3}$$

If the straight line passes through the origin O, like OA, then

$$\frac{y}{x}=c \tag{4}$$

or the variables are directly proportional to each other. If then, the empiric curve is a straight line, the function will be one of the above equations. If both variables are zero at the same time, then (4) will be the true case. As example construct your work in El. Phys. 41, and 94–95, here taking p as abscissa and r as ordinate.

15. A parabolic curve like OD, passing through the origin—or having y=o for x=o—corresponds always to

$$\frac{y}{x^m} = c \qquad (5)$$

or the dependent variable y is proportional to the m-TH power of the independent variable. If m=2, the curve is the common parabola; if m=3, it is called the cubic parabola. The greater m the more steep the ascent of the curve; this will therefore guide in the guess for m, for which different values have to be tried until the true one results. For examples, construct your work El. Phys. 48, also 43, which will give common parabola; also El. Phys. 44 to obtain a cubic parabola.

16. A hyperbolic curve, like EF (Fig. 5), has asymptotes or branches which more and more approach a straight line without ever completely reaching the same. Thus the axes OX and OY are asymptotes for the branches F and E if the equilateral hyperbola drawn is Fig. 5, which is represented by

$$x.y=c \qquad (6)$$

that is, the product of the two variables is constant; or, what is the same, because (6) also may be written

$$y=\frac{c}{x} \qquad (7)$$

the dependent y is inversely proportional to the independent x. For examples construct your work in El. Phys. 255–256, by taking o as abscissa and i as ordinate. You will then obtain equilateral hyperbola, but you must subtract the focal length f (obtained from the mean of all observed values $\frac{1}{o}+\frac{1}{i}$) from each, so as to have o—f=x and i—f=y in order to find (6) satisfied.

Such a change evidently corresponds to the adoption of the point having f as abscissa and ordinate in the original drawing as the new origin for x and y.

17. As second example experiment with the lever (El. Phys. 94–95) by taking both R and r constant, which will give the momentum of the resistance R.r=constant. Then carefully determine P and p by balancing the above momentum; record and construct as above directed, taking p as abscissa x, and P as ordinate y. You will then find the empirical curve to be an equilateral hyperbola.

To obtain accurate values, suspend R by a fine silk cord, and take R about 100 to 200 grams, r about 30 to 40 cm., so that Rr is about 6,000 gr. cm.

18. If now the empirical curve resembles one of the above described three algebraic curves (or geometrical representations of algebraical functions); then it remains next to ascertain, whether the curve suggested is the true one; that is, whether the corresponding function accurately expresses the relation between the observed values of x and y. To test this we proceed as already stated in El. Phys. 41 and 48.

In a fifth column of the table begun in art. 10, put the calculated value of the quantity q which is to be constant if the assumed curve is the true one. Thus if the equilateral hyperbola appears to be the curve, then try it by calculating the products q=x.y for each determination; for according to equation (6) in art. 16, these values should be constant =c; the values x y are accordingly entered in the fifth column, and if they are nearly equal, the mean m of all the values in this column is taken as the most probable value of c. In general, the mean m of all these values q entered in the fifth column is taken as the most probable one and called c. Of course, if the values in this column differ very much, no mean should be calculated, because then the

assumed relation or curve is proved to be false; in that case some other relation will have to be tested, until the true relation is found.

19. In no case will these values q of c calculated for each of the determinations made be quite equal. On account of the unavoidable errors (which will be considered in detail further on) the different determinations cannot completely agree (compare your work in El. Phys. 41 to 48; etc.). It is, therefore, very important to have some definite measure of the probable error in the final result m for c; so that we would know the true value of c to be most probably smaller than m+e and greater than m—e; in other words

$$c = m \pm e \tag{8}$$

The amount e is often called the uncertainty of c; for the smaller e, the more accurately the value of c is known.

20. This probable error e is calculated in the following manner (see School Laboratory, 1871, p. 94). For each determination find the difference* between the mean value m and the actually observed value q, that is

$$d = m - q \tag{9}$$

which enter in a sixth column; in the seventh column enter the square of these differences, d^2; finally take the sum S of this column. Then the probable error e of the mean m will be

$$e = \frac{2}{3}\sqrt{\frac{S}{n(n-1)}} \tag{10}$$

which numerical value substitute in (8).

21. If more than one constant is to be determined then the above method is not applicable. If only two constants, c and k, are to be determined, and

* Also called residual.

if the equation containing these constants is of the first degree in regard to these constants, then they can be readily determined in the following manner* :

Let X Y Z represent any functions of any two or three quantities, which are determined by experiment (as x and y above); then if the final relation is of the form

$$X+cY+kZ=o \tag{11}$$

it can be reduced to

$$\eta=c+k\xi \tag{12}$$

by division, etc., so that all values of η and ξ are numerical and can readily be calculated for the determinations made. This equation represents a straight line (see art. 14). Hence enter ξ as abscissa and η as ordinates, draw a straight line through these points, deviating everywhere as little as possible from the points determined: this line will cut off the value of c from the vertical axis, and give k as the increase of the ordinates for an increase of a unit in ξ. Of course, it is best to take the increase i of η for an increase of 100 units in ξ, because it can be determined more accurately; then $k=\frac{i}{100}$.

If the scale was such that the vertical axis is not on the drawing, substitute this value for k in (12) and solve the equation for any proper point in the line.

22. A simple example of this kind we have in the experiments described in the School Laboratory 1871, pp. 105–108, for the demonstration of Mariotte's or Boyle's Law, in case the height of the barometer, b, should be unknown (as in case there is no barometer at hand). Then the pressure p=b+h contains the unknown constant b; the volume v and difference in level h be-

* This most practical geometrical solution of the rather intricate method of the least squares for this case, will be found useful by others than students.

ing determined* as before, we have to test the observations by the equation of the equilateral hyperbola

$$p.v \text{ or } (b+h)\, v = c \tag{13}$$

which gives

$$h.v = c - b.v \tag{14}$$

which is of form (11). By making $h.v=\eta$ and $v=\xi$ we have immediately (12), $k = -b$.

23. After the most probable value of the constant† c with its probable error e (see 18–20) has been determined, the function (1) is completely known, so that for any given value of the independent x we can calculate the most probable value of the dependent y which corresponds thereto. Each of these values of y thus calculated evidently rests upon all the experiments made, because the value of the constant used in the calculation was determined from the entire number of observations.

24. We can therefore calculate the value y' of y which corresponds to each of the used values of x; the values y' thus calculated are entered in the eighth column.

$$y' = c.f(x) \tag{15}$$

25. Since these calculated values are the most reliable (see 23), we may next calculate how much the observed values y are in error; or what correction δy has to be applied to the observed value y in order to give the calculated value y', that is

$$y' = y + \delta y \tag{16}$$

consequently

$$\delta y = y - y' \tag{17}$$

These corrections are now entered in the ninth column of the table; carefully noting the SIGN (positive or negative) of the same as determined by (17).

*Except that it is not necessary now to reduce the observed height h' to mercury.

† Or the two constants by 21.

26. These corrections are finally entered in the graphical representation of this series of experiments (see 11). But as they are necessarily comparatively small, they should be entered with a scale 10 (or more) times as great as the scale used for y (state the scale actually used). And in order to obtain room for the negative values of these corrections, some of the heavy (red) lines near the middle of the page should be used as axis of abscissa for these deviations. As a matter of course, they are entered for each abscissa x used. The extremities of the consecutive deviations are joined by a fine, dotted, straight line. Hence the line of deviations will be a broken one.

27. The magnitude of these corrections will now directly exhibit the degree of accuracy of the work done and the degree of confidence that may be given to law established by means of the observations and reductions.

These corrections should be partly negative and partly positive. If also in quantity less than the possible error of a determination (see School Laboratory, 1871, p. 83–93), then the law may be considered fully established to the degree of accuracy of the experiment used. That is, if there is any deviation from the law, it is less in amount, than the possible errors of the experiments made.

28. It may be well now to recapitulate the order of the work to be done in each single series of experiments of inductive physics. We shall refer to the table here added (page 19), containing all of the 9 columns mentioned in the preceding. For the sake of convenience we have on this blank table inserted the number of the column in the third line, and in the fifth line the number of the article where each column is first referred to. The Result should of course be filled out in proper manner; c numerical, y just as found with proper value of c (mean).

SUBJECT..

No.	OBSERVATIONS.			REDUCTIONS.					REMARKS.
	r	x	y	q	d	d^2	y'	δy	
1	2	3	4	5	6	7	8	9	(Column.
Art.	10	10	10	18	20	20	24	25	(Article.)

mean, $c =$ .... $S =$ ....

$e =$ ....

Result: $c = m \pm e$ $y = c.f(x) =$

If, however, several reductions have to be made in order to obtain the value of q, it is well to insert a corresponding number of columns preceding q, under the head of "reductions." Also, if r is the same throughout, then it may be so stated in words, and the column r may be omitted.

The order of procedure then is:

1. Study the subject given: see whether evidence has been given that y depends on x only.

2. Construct table, but drawing only enough verticals for the observations.

3. Make the observations most carefully, and enter the same (see art. 10).

4. Enter the same on chart (see 11), and draw the empiric curve as carefully and elegantly as possible.

5. Find which curve it resembles (usually directed by teacher). Accordingly, you will know what reductions are to be made, and hence how many (if any) columns are to precede q. Carefully make these reductions, enter the values q, and take the mean m therefor. Be very sure that your calculations are right! Hence, repeat several, or all, of them, and test them in other manners also.

6. Calculate the differences d, and their squares. Be sure not to err in regard to decimal place! For simplification of this work, see School Laboratory, 1871, p. 95.

7. Take the sum S of these squares, from which calculate the probable error e of the mean.

8. This will now enable you to write the final result (e and y) at the bottom of the table. Write plainly!

9. Now calculate, from this formula of y, the values y′ which enter in table.

10. Calculate corrections δy of observations, and enter the same.

11. Enter these corrections on chart to proper scale (see 26).

12. Special remarks, plain and concise, may have been entered at proper place in table.

If any observation is known to be faulty, it should be entered anyhow, for the table is to be a faithful record of the work done. But such observations should n o t be reduced — that is, all columns, from q on, should be left blank opposite such entries.

29. In this manner, several of the series of experiments of the E l e m e n t s o f P h y s i c s should now be carefully and completely reduced — at least o n e case in each of the articles **14, 15, 16.**

We may also add, for the straight line (art. 14), one of the series of experiments with pulleys (El. Phys., 90, 91), where —

$$\frac{P}{R} = c$$

and c will be found equal to the reciprocal of the n u m b e r of supporting strains of thread, plus a small fraction expressing the friction.

The law of the lens (El. Phys., 255, 256), which comes under art. 16, as a hyperbola, may also be worked thus:

$$\frac{o.i}{o+i} = c$$

which equation is equivalent to the one in El. Phys., 255, if $c = f$. Here two columns would precede q in our table — the first would contain the sum $o+i$; the second, the product $o.i$.

30. In concluding this short exposition of the general elementary method of quantitative physical induction, we once more repeat that it is necessary for the student to be scrupulously careful and accurate in his observations and experiments, as well as in his drawing, recording, and in all calculations. An error in either link breaks the chain of evidence and destroys the work — the labor of hours or days!

III. ON THE DEGREE OF PRECISION.

31. The probable error e of the mean value m of the constant c indicates the degree of precision wherewith the value of this constant has been determined, p r o v i d e d no constant errors affect the experiments (see School Laboratory, 1871, pp. 96–98). The c o r - r e c t i o n s (see 25 to 27) of the single experiments also measure the degree of precision of the determinations made. How all of these quantities depend on the p o s - s i b l e e r r o r (School Laboratory, 1871, pp. 88–93), has already been referred to (in 27).

32. Now it is in this d e g r e e o f p r e c i s i o n that the progress of quantitative physical science is expressed, whether we look to the advance of science itself, or to the p r a c t i c a l mastering of acquired portions of sci-ience by the progressing student. While to the student of the r u d i m e n t s of physics, gram and centimeter are the smallest magnitudes commonly used, the student of the e l e m e n t s must determine the first decimal and bring his errors usually inside the second decimal of these units, and the student of the p r i n c i p l e s very frequently determines the second decimal direct, thus bringing the errors of his results down to and below the third decimal of the gram and centimeter. Of course, in some experiments a higher degree of precision is attained, even at a low stage of the student's advancement.

33. Finally, in the case of the special student and investigator, a much higher degree of precision is required, and, in most cases, attained, by means of i n s t r u - m e n t s o f p r e c i s i o n, and by the substitution of s e l f - r e g i s t e r i n g and a u t o m a t i c instruments for the personal organism of the experimentor. The latter devise, of course, eliminates most p e r s o n a l errors —for which the usually much smaller and more constant instrumental errors are substituted.

34. Whenever such higher degrees of precision are aimed at, each determination must be separately r e d u c e d, by eliminating, as much as possible, the disturbing causes from the same. Thus, every LENGTH measured must be corrected or reduced, first, by allowing for the e r r o r o f t h e s c a l e used, an error which has been once for all determined by comparison with the standard; and, second, by allowing for the e x p a n s i o n of the scale, by observing the temperature at which the measurement is taken. So, also, every VOLUME measured must be corrected for standard and expansion. Every WEIGHT determined must be corrected for error of the weights used, and for varying b u o y a n c y of the air—varying with temperature and pressure. For this reason, very careful weighings are reduced to a b s o l u t e w e i g h t—that is, the weight which the body would possess if weighed in a perfect vacuum. So, also, every measure of t i m e must be corrected for its disturbing elements.

In all these cases, the table (28, 3) should contain, besides the value directly observed, also the corresponding corrections and the reduced values in as many additional columns. Only the last values are, of course, to be entered graphically as co-ordinates, and to be used in the reduction of the series of observations.

35. For the use of students in the Principles of Physics, the reductions given in the subsequent articles are quite sufficient.

The v a l u e s of the corrections ε will always be given for one unit of the quantity q determined. Hence the total correction will be $q\varepsilon$, and the corrected value will be

$$q' = q + q\varepsilon = q(1+\varepsilon) \tag{18}$$

Further, in all calculations bear in mind that these corrections of the unit are very small, so that their products and higher powers are practically utterly insignifi-

cant, and therefore may be omitted. Thus, if the square of q′ were calculated, it would simply be $q^2(1+2\varepsilon)$. See modes of calculation, with small corrections (or errors), in School Laboratory, 1871, pp. 89–91.

36. CORRECTION FOR EXPANSION.—The expansion of a unit for one degree centigrade is called the co-efficient k of expansion. (Compare El. Chem., 40.) The following co-efficients of expansion are most used:

Expansion in length:

Glass	8 millionths.
Brass	19 millionths.
Steel	39 millionths.
Mercury	180 millionths.

Expansion in volume:

Glass	25 millionths.
Mercury	180 millionths.
Water	500 millionths.
Air	$3\frac{2}{3}$ thousandths.

To apply these corrections, multiply these co-efficients k with the number n of degrees centigrade for which reduction has to be made, and this product with the value q observed; then q.n.t will be the correction, which must be added to or substracted from the observed value q, according as the reduction is to a higher or lower degree of temperature. Work examples!

For the reduction of the height of the barometer to the freezing point, special tables are more practical.

37. Volumes of gases are reduced by the so-called law of Boyle (Mariotte)-Gay Lussac. Let v be the volume observed at a temperature t°C and pressure p; let it be desired to reduce the volume to a pressure p′ and a temperature t′; then this reduced volume v′ will be determined by

$$\frac{v'.p'}{T'} = \frac{v.p}{T} \qquad (19)$$

or
$$v' = \frac{T'}{T} \cdot \frac{p}{p'} \cdot v \tag{20}$$

where $T=273+t$, the absolute temperature of observation, and $T'=273+t'$, the absolute temperature of reduction. Work examples!

Boyle's or Mariotte's Law is (19) for $T=T'$; it is obtained by induction, as directed, School Laboratory, 1871, p. 105.

38. Correction for Buoyancy, or Reduction to Vacuum.—At common temperature (17° C) and pressure —usually 750 mm. in common altitudes (see El. Phys., 130),—one cubic centimeter of dry atmospheric air weighs 1.2 milligrams, or twelve ten-thousandths of a gram. (Compare El. Phys., 33.)

Consequently, any body having a volume of v cubic centimeters, suffers a buoyancy equal to

$$\partial w = \frac{12\ v}{10,000} \tag{21}$$

grams, which, added to its apparent weight, will give its absolute weight, or weight in vacuum.

If the volume of the body cannot be directly determined (El. Phys., 44, 47), it may be found from the apparent weight w and the specific gravity G of the same; for, by El. Phys., 47, we have

$$v = \frac{w}{G} \tag{22}$$

Here G must be determined by El. Phys., 123—that is, by finding the weight w′ of the body submerged in water; then:

$$G = \frac{w}{w-w'} \tag{23}$$

Since larger weights are usually of brass—and for small platinum weights no reduction need be made—of a specific gravity $G=8.4$, the volume of brass weights weighing w is, by (22),

$$v' = \frac{8.4}{w}$$

giving a buoyancy on the side of the weights, by (21), equal to

$$\delta w' = \frac{1.43}{10,000} w \tag{24}$$

Hence, if a body is weighed by brass weights, its weight w will be reduced to vacuum by adding the correction—

$$\delta w'' = \frac{12 v - 1.43 w}{10,000} \tag{25}$$

where v the volume, w the apparent weight of the body determined by brass weights.

For lead weights, $G = 12$ very nearly, (25) becomes

$$\delta w'' = \frac{12 v - w}{10,000} \tag{26}$$

These corrections being so small, slight variations in pressure and temperature of the air are of no appreciable influence. If, however, they are to be allowed for, it can be done by proper use of art. 37.

39. CORRECTION FOR ERRORS IN SCALE AND WEIGHTS. — After the above corrections have been applied, the corrections for scale (length, volume, etc.) or weights should be applied, if of sufficient amount. Of course the value of these corrections can only be ascertained by a careful comparison with good standards. Thus the standard meter at the Laboratory of the Iowa State University has,* at 0° C, between the cross-lines 0 and 100 cm., measured midway between the lengthwise lines marked 4 and 5, a length of

1 meter + 0.042 mm.

or the correction of this meter rod is plus 42-millionths.

* According to the comparison kindly made by J. E. Hilgard, United States Coast Survey, Washington, in March, 1872.

The Hectogram standard weighs, according to the same authority,

100 gr. + 0.2 mgr.,

being a correction of 2-ten-millionths of the total.

Weights and measures actually used by students are compared with these standards of the Laboratory, and the corrections given to stúdents, to be applied whenever necessary.*

40. The instruments of precision used in modern physics are far too numerous for description here. Many such instruments will be found described in any of the larger treatises on experimental physics (Müller, Wüllner), but especially in serial publications for research in physics — such as Poggendorff's Annalen, and the Annales de Physique et de Chimie, and the publications of the great learned societies† of Europe. The Repertorium der Experimental Physik, edited by Dr. Carl, of Munich, is almost exclusively devoted to such descriptions.

Here it will be quite sufficient to refer to some of the principal devises most extensively used in the construction of such instruments.

41. For the accurate determination of minute divisions on any scale, the latter is provided with a vernier, which slides along the main scale. The simplest form of a vernier is a small scale, wherein nine parts of the original scale are divided into ten equal parts. Hence the difference between one division on scale and vernier is $^1/_{10}$ of a scale division.

The use of the vernier is best learned by actual practice; for example, by reading a barometer and a goniomater provided with vernier.

* We shall with pleasure compare measures and weights for teachers, provided they pay expressage both ways.

† Especially the Sitzungsberichte der Kais. Akademie der Wissenschaften, Wien.

42. The micrometer screw is used for measuring very small lengths with precision. A very accurate screw, having a thread say $^1/_2$ mm., provided with a head divided into 500 parts, will enable us to measure to the thousandth of a millimeter. Such micrometer screws are attached to various instruments, especially to microscopes and telescopes.

43. Telescopes, which are attached to divided circles for the measurement of angles, are always provided with cross-wires at the distance from the eye-glass where the image is formed by the object-glass. A circular hole in screen T (School Laboratory, 1871, Plate I., Fig. 1), provided with cross-hairs, will exhibit the working of this important devise for the accurate determination of the line of sight.

44. To measure very small angles the method by reflection is generally used. A mirror is attached to the object, small angular motions of which are to be measured. At some distance (from 1 to 5 metres) therefrom a fixed scale and a telescope are placed. The division of the scale is read by the telescope. This method of reading corresponds to the use of a graduated arc of a radius twice as great as the distance from the mirror to the scale.

Gauss applied this method to the observation of the variation of the magnetic needle (El. Phys. 339). In very delicate galvanometers (El. Phys. 409) the magnet is a minute mirror of steel; a beam of light is thrown upon it—and the minute motions of the needle-mirror become visible in the great motion of the reflected image. Such galvanometers are used in connection with the Atlantic cable (El. Phys. 415); also for the purpose of feeling the pulse of a patient some hundreds of miles distant.

45. The galvanic current and its temporary magnetizing effect on soft iron (El. Phys. 420–423) is very

much used in all sorts of self-registering instruments and in chronoscopes. A Chronoscope is an instrument to measure very short intervals of time; for example, in hundred-thousandths of a second! By means of such instruments the motion of a projectile in the barrel of a gun, produced by the explosion of the charge, is accurately followed, although the duration of this motion* is less than the one hundredth of a second of time!

46. Of self-registering instruments, modern Physics possesses the most surprising contrivances, from those fixing every vibration of the fleeting sound permanently on paper† to instruments which give a permanent graphical record of the state of the atmosphere,** or even read the barometer and immediately print*** the number read off!

47. By the use of instruments such as referred to in the preceding articles, modern science becomes daily more independent of the imperfections of our own organization. By the telescope our feeble eyes are made to reach far into cosmical depths; by the microscope they are made to distinguish the most minute: by the spectroscope they see into the very chemical constitution of

* In Captain Noble's lecture on the explosive force of gunpowder, in Revue Scientifique, 1872, No. 48, p. 1136. Also General Morin, in Comptes Rendus, 1872, T. 74, p.841, where the observed intervals of time from the ignition of the charge of 13.6 kilograms powder back of a cylindric projectile of 81.6 kgr. are given in one hundred thousandths of a second for 12 distinct points in the barrel of a gun 2.2 metres in length! Thus the motion of the projectile in its course along the gun barrel is as well known as the motion of the minute hand over the dial of a clock! See also Scheebeli in Zuerich Vierteljahrsschrift, 1870, p. 257, for the determination of the duration of impact of two steel bars.

† See Phonautograph in more recent treatises of physics: also Pisco Die neueren Apparate der Akustik, Wien 1865; especially pp. 55–93.

** Of such instruments there are many now in use. See Secchi's in Ganot, Physique, 13ed. Paris, 1868, pp. 865–874. Also Hough's, and others, in Nature 1872, Vol. 4, pp. 390, 410, 430.

*** Hough's Typo-Barograph. Annals of the Dudley Observatory, Vol. I, 1871. Also Nature, l. c.

bodies, whether near or far! So by the chronoscope there is no phenomenon so fleeting but we may leisurely trace its course; whether it be the motion of the ponderous projectile moved along the barrel of a gun by the explosive force of gun-powder, or the silent act of deliberation* in the human brain — the phenomenon has in either case been accurately traced in time and speed by the chronoscope.

48. If there remain residual errors after all proper reductions of careful experiments have been made — and, especially if these residual errors follow some definite order: then they do indicate some disturbing cause overlooked in the reduction of the experiments. The new problem then arises, to find the precise nature of this cause. The deviations produced by the same, or this kind of residual errors, are often very properly called **perturbations**.

Thus experiments with gases have shown that in the neighborhood of the point of condensation all gases deviate from the law stated in 37; in that case the perturbations observed are of course known to be due to the approaching change to the liquid form.

In the same manner the motion of the planets has been found to differ slightly from that which is determined by Kepler's Laws. These deviations or perturbations are accounted for by the mutual attraction of the planets. The residual systematic errors in the motions

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* The reflection, whether a certain impression was produced on the right or left hand required in a given person 66 thousandths of a second (with a probable error of 0.004 seconds). The reflection necessary to decide between red and white was 154 thousandths second; to reflect which vowel was heard 88 thousandth second. See de Jaager's Experiments, made in Donder's Laboratory at Utrecht; Fortschritte der Physik 1866, pp. [illegible]; Also Der Naturforscher, [illegible] pp. [illegible] — See Exner, Sitzber. Wien, 1868, II, 58, 601–632, and compare Exner's determination of the intensity of the impression on the retina at ten different moments from 8 to 166 thousandths of a second (l. c. p. [illegible]) with the determination of the space passed over by a projectile of 82 kgr. in the barrel of an English gun at 12 distinct intervals of time from 1 to 10 thousandths of one second (foot note to article 45).

of the planet Uranus, remaining after the perturbations due to all known planets had been allowed for, pointed directly to the existence of an additional planet never seen by eye; this planet was determined from the residual perturbations, and thereafter discovered by the telescope: the planet Neptune!

49. Precisely this mode of induction from residual errors or perturbations is constantly practiced in Physics proper. The great value of the results thus obtained indicates that the study of the perturbations is of exceeding importance to inductive physics. See Hinrichs' Contributions to Molecular Science, I to IV; also our Atomechanik.

50. If the student carefully reviews—and as far as possible applies—the contents of this section on the degree of precision, he will become convinced that inductive physics is properly called an exact science, not only because the errors actually committed are so excessively small, but especially because a higher limit of the value of these errors can always be ascertained. In other words, the results stated are positively known to be true inside of accurately determined limits, which, when the methods and instruments of precision are properly used, we have seen to be exceedingly minute.

IV. INDUCTIVE DYNAMICS.

51. The principal laws of dynamics can be inductively established (compare 6). These inductions form not only excellent examples of experimental induction for student's practice of sections II and III, but the laws thus established are constantly used in all other branches of physical science for the explanation of phenomena.

Instead of giving a full exposition of inductive dynamics, as we would like to do, we must limit ourselves to an enumeration of the principal series of experiments

to be performed and the statement of the result to be obtained. All other detail must be supplied by the teacher at the Laboratory. It is very convenient to have a concise exposition of these details written on cards, which are numbered in accordance with the articles here printed, and given to the student when about to perform the experiments.

1. UNIFORM MOTION.

52 Space s equal velocity v into time t.

$$s = v.\,t \tag{27}$$

53. Water clock. See School Lab. 1871, p. 99. Find volume u per second.

54. Walk on level road (plank walk); distance d metres, time t seconds, number of paces n. Find time of one pace τ and length of one pace l

$$c = \frac{d}{t} \qquad l = \frac{d}{n} \tag{28}$$

Find formula for distance d determined by time; to be used in practice.

2. UNIFORMLY ACCELERATED MOTION.

55. Galileo's Experiment. Length of incline s down which the ball rolls in time t (reduced to seconds, observed on clock beating about $\frac{1}{2}$ or $\frac{1}{3}$ seconds). Result,

$$\frac{s}{t^2} = c \tag{29}$$

From which for uniformly accelerated motion generally

acceleration, $a = 2c$ (30)

velocity. $v = at$ (31)

space, $s = \frac{a}{2} t^2$ (32)

also from (31) and (32), $v^2 = 2as$ (33)

56. Same experiment with water clock. See School Lab. 1872, p. 9.

57. Atwood's Experiment. Use two cylinders with water clock—in first collect water during accelerating motion down space s; in second during uniform motion down space s′. Both cylinders held, close together, in right hand. Change s′ several times for same s (compare 52). Even without friction wheels very good result.

58. Free fall. State laws. Acceleration a = g = 9.809 metres.

59. Mechanical work W and actual Energy E. Weight of a body w raised to height h requires work W = wh (El. Phys. 84).

Same body descending down same height acquires velocity v. By (33)

$$W = w.h = \frac{mv^2}{2} = E \tag{34}$$

So that mass m of any body

$$m = \frac{w}{g} \tag{35}$$

The product mass into square of velocity is often called vis viva. Solve problems; also the one indicated El. Phys. 100, at close.

60. Problem. Demonstrate for any acceleration the corresponding formulæ, and compare to El. Phys. 97, 450; El. Chem. 80, 98, 162, 163.

3. MOTION OF PROJECTILES.

61. Orbit of projectile is a parabola. Measure height y′ of water jet at different x from orifice; also height of orifice k; then y = k — y′ fall. You find (see 15)

$$\frac{y}{x^2} = c \tag{36}$$

62. How will resistance of air modify the orbit of a round projectile? Rifles, rifled cannon.

4. LAWS OF PENDULUM.

63. State. Demonstrate in regard to amplitude and material in lecture; in regard to length l and time t by (School Lab. 1871, p. 103)

$$\frac{l}{t^2} = c \qquad (37)$$

64. Calculate

$$g = \pi^2 c \qquad (38)$$

65. Number of vibrations n in five minutes measures the intensity of the accelerating force f; for (37) and (38) give

$$f = c.n^2 \qquad (39)$$

66. Demonstrate that intensity f of action of a magnet pole on pole of small magnetic needle is at distance a (El. Phys. 318; 313).

$$f = \frac{c'}{a^2} \qquad (40)$$

Number of vibrations of needle under influence of terrestrial magnetism e alone r; under joint influence of f and e the number be n; then you find, as expression of the law (40)

$$(n^2 - r^2) a^2 = k \qquad (41)$$

67. COMPOUND PENDULUM.—Time of oscillation t, statical momentum S, inertia momentum I, then length l of corresponding simple pendulum

$$l = \frac{I}{S} \qquad (42)$$

which is demonstrated by (37),

$$\frac{I}{S.t^2} = c \qquad (43)$$

$$I = \Sigma mx^2 \text{ and } S = \Sigma mx \qquad (44)$$

being calculated from the known masses m and their distances x from the axis of rotation. Here weight w may be used instead of mass; compare (35) and (43), (44). Apparatus used: pine rod, 25 dm. long, 5 cm. wide, 1 cm. thick, and lead weight of tabular form, sliding along the rod, supported by wooden pegs.

68. L i m b s o f a n i m a l s as compound pendulums. Determine time of oscillation of your leg by counting number of oscillations in ten minutes—while standing with the other foot on a block 1 dm. high. Compare result to 54. Brothers W e b e r found same value as time of pace. What importance of this result in mechanical work of walking and of marching?

5. ROTARY MOTION.

69. It is easy to demonstrate that the actual energy of rotation is

$$[illegible] \tag{45}$$

if ω a n g u l[illegible] that is, the linear velocity at the unit of dista[illegible] the axis, and I inertia moment (44).

70. F l y - w h e e l. If weight of rim w kilograms, distance of middle of rim from axis r meters; then, by (44) and (35)

$$I = \frac{w}{g}.r^2 \tag{46}$$

If number n of rotations per minute, then you find

$$\omega = 0.10462.n \tag{47}$$

and

$$E = \frac{w.r^2.n^2}{1789} \tag{48}$$

The actual energy of the fly must be k times the mechanical power M of the machine per second. In common machines k = 30; in machines which are to have

a very uniform motion, k is often as high as 60. For a rolling mill with 60 horse power machine, the fly has r = 3 meters and revolves once a second, k = 30 will require how heavy a fly?

71. Radius of gyration R is the distance from the axis at which the entire mass M may be supposed collected; hence

$$I = M.R^2 \tag{49}$$

A globe of radius r rotating around a diameter R = 0.632.r. A cylinder of radius r, rotating around its geometrical axis, R = 0.707.r. A rod of length l rotating around an axis at right angles to its end: R = 0.289 l.

Problem: Chassepot ball has: velocity 420 metres, revolves 800 times a second, is 22 mm. in diameter, and 26 mm. long, weighs 25 grams. How great its actual energy of translation E and of rotation E′, in kilogrammeters? Also the ratio of E′ to E.

72. Centrifugal Machine. Demonstrate by proper experiments, that an axis of rotation of any point of a body for which moment of inertia is greatest is most stable.

Theory shows that the axis for which I is a minimum is also a stable axis of rotation; and that it is right angles to the former.

These two axes are therefore permanent axes of rotation.—If they pass through the center of gravity, the axes are called natural axes, and need not be supported unless forces act from without on the body.

The celestial bodies all revolve around that natural axis for which the momentum of inertia is a maximum.

The atoms of gases revolve around this same natural axis of maximum momentum of inertia; the atoms of liquids revolve around that natural axis, for which the momentum of inertia is a minimum (Hinrichs).

73. Experiment with Plateau's Machine, illustrative of cosmical phenomena according to the Nebular theory.

74. Experiment with Gyroscope to prove that two axes of rotation give a resulting axes of rotation in the same manner as two forces give a resultant (El. Phys. 103–110).

In the absence of a gyroscope a common top, or the tupie a friction, will exhibit some of the phenomena.

6. FREE MOTION OF A BODY.

75. Experiment with the card-paper boomerang in the Laboratory, to exhibit the various laws of motion. Excentric impulse; motion of translation, rotation around most stable natural axis; screw motion on account of resisting medium; causing the boomerang to return to the starting point after having described a most elegant curve of double curvature in the room. The boomerang need only be 9 cm. long, 1 to 1.3 cm. wide, weighing about half a gram; the angle subtended should be 100 to 110 degrees, and the plane of the paper should be bent about 5 degrees at the elbow.

V. APPENDIX.

76. Innumerable phenomena of nature are periodic — returning after certain equal intervals in precisely the same intensity. The simplest case of this kind is exhibited in the motion of the pendulum — all phases of this motion, as to place, velocity, etc., are strictly periodic. The phenomena of day and night, summer and winter, in fact all phenomena depending on the motion of sun and moon are equally periodic. The motions of the vibrating string, the wave motions in water, even light and heat are periodic in their very nature. No algebraic curve (see 13) can express such phenomena; the very essence of these phenomena refers us to the circle as the means of expression of the same.

Here it must suffice to explain the nature of the circular functions so far, that the student may understand how the values of these functions are obtained for any given value of the independent x; and, therefore, how the tabulated values can be used for inductive research in physics.

77. The arc of circle is taken as independent x; for example, arc AB, Fig. 6. If the corresponding angle be a degrees, then

$$x = \text{arc } a$$

Since all circles are similar (El. Phys. 48, closing remark), we need only consider the values of the circular function (such as arc) for the radius $= 1$; if the actual radius be r, then the corresponding values of the functions will be obtained by multiplication with r.

It will be noticed that the arc increases indefinitely in the direction A B, etc. The arc of a semi-circle is $x = \pi = 3.1416$ (El. Phys. 48), of 3 circumferences $x = 18.8466$, etc., etc., without limit — thus exactly corresponding to the unlimited increase of time in motion. The point A is taken as the initial point, or the beginning of all arcs; they are always measured from A in the direction ABDEF AB, etc., etc. If the end B of the arc is between A and D, the arc is said to be in the first quadrant ACD; if the end of the arc is in DE, the arc is said to be in the second quadrant, etc., as indicated by the roman number in Fig. 6.

For convenience, we suppose this arc unrolled along the axis of abscissa in Fig. 7 — the distance ADE being exactly equal to the corresponding arc ADE in Fig. 6.

78. The circular, or more properly the trigonometrical functions most used are the sine, cosine, tangens, and cotangens of the arc x; these are commonly written

sin x cos x tg x cotg x

To any arc x corresponds an accurately determined value of each of these functions, readily obtained by direct construction in the following manner:

Let the radius unity of a circle, Fig. 6, be taken sufficiently large, for example, CA = 1 decimetre = 100 mm. Then for any angle BCA = a the length of the lines drawn as stated below will give the value of the trigonometrical functions constructed, with sufficient accuracy for many purposes:

The sine of angle a or arc x is the line BP drawn from the extremity B of the arc to the fixed diameter ECA of the beginning of the arc.

The sine is therefore always less than the radius (unity), and positive if drawn down to EA, and negative if drawn up to EA.

The ordinates in the curve AdEfG are the sines of the arcs which are abscissa; hence this curve is the sine-curve or SINOID.

The cosine CP of any arc x is the line cut off by the sine on the fixed diameter from the center of the circle to the sine; hence positive to the right, negative to the left of C. Fig. 7 the line a De Fg is the consinoide.

The tangens AT of any arc x = AB is the piece cut off by the prolonged diameter of the extremity B from the fixed tangent tAt′ of the beginning A of the arc; the tangens is therefore positive up from A towards t′, negative from A down towards t. The curve Ad′, d″Ef′, f″G represent the tangentoid; its ordinates are equal to the tangens of the arc as abscissa.

The cotangens DQ is in the same manner cut off on the fixed tangent to D, parallel to the fixed diameter ECA. The dotted line a′De″, e′Fg″ represents the cotangentoid.

79. The exact value of each of these functions for any given angle or arc has been most carefully deter-

mined and tabulated in the so-called trigonometrical tables. For most physical investigations the student will need tables giving these values with three or four decimals only. Page 3 of the table of J. N. Peirce, published 1871, by Ginn Brothers, Boston, will answer very well.

The following values should be verified by the student's own construction; a drawing with a radius half a decimetre, Fig. 6, will suffice. By means of these values the student should also construct the curves, Fig. 7, to scale unit = 2 cm. or more.

TABLE OF TRIGONOMETRICAL FUNCTIONS.

RADIUS = 1,000.

angle a	arc x	sine sin x	cosine cos x	tangens tg x	cotangens cotg x
0	0.000	0.000	1.000	0.00	∞
30	0.524	0.599	0.866	0,58	1.73
60	1.047	0.866	0.500	1.73	0.58
90	1.571	1.000	0.000	∞	0.00
120	2.094	0.866	—0.500	—1.73	—0.58
150	2.618	0.500	—0.866	—0.58	—1.73
180	3.142	0.000	—1.000	0.00	∞
210	3.666	—0.500	—0.866	0.58	1.73
240	4.189	—0.866	—0.500	1.73	0.58
270	4.713	—1.000	0.000	∞	0.00
300	5.236	—0.866	0.500	—1.73	—0.58
330	5.760	—0.500	0.866	—0.58	—1.73
360	6.284	0.000	1.000	0.00	∞

80. While we cannot in this introduction enter upon the application of these functions to any of the phenomena referred to in 76, we shall show how some of the laws demonstrated in the Elements of Physics may be expressed by trigonometrical functions.

It is evident that if AC, Fig. 20 (El. Phys. Plate 1), be taken as radius, CD will be the sine of the angle $a = CAD$, the inclination of the plane to the horizon. Hence the law of the inclined plane (El. Phys. 93) becomes

$$\frac{P}{R} = \sin a \tag{50}$$

Which verify by means of the tables of natural sines, or by construction.

The parallelogram of forces (El. Phys. 106 to 109; Fig. 19) may be expressed trigonometrically, thus

$$\frac{P}{\sin(\overline{Q.R})} = \frac{Q}{\sin(\overline{P.R})} = \frac{R}{\sin(\overline{P.Q})} = c \tag{51}$$

which you also may test by your observations, if you have a table of natural sines—taking of course the observed angles as well as the observed forces. The above three quotients should be equal for each experiment separately.

81. The logarithmic function is represented in Fig. 8; the ordinate y being the "logarithm of x," which is written

$$y = \log x \tag{52}$$

x being that number which results if 10 is raised to the y^{th} power, or

$$x = 10^y \tag{53}$$

The number is abbreviated "num." (numerus). The following simple values are easily obtained:

log	0	1	2	3
num.	1	10	100	1000

82. The precise value of the logarithms of all numbers has been carefully calculated and tabulated in special logarithmic tables. Pages 4 and 5 of Peirce's Tables are quite sufficient for student's use.*

* They may be obtained, printed on strong card paper, by addressing the publishers, Messrs. Ginn Brothers, 3 Beacon street, Boston.

For convenience, the logarithms of the trigonometrical functions are usually printed with the logarithms of the numbers. The trigonometrical functions are then distinguished as the natural sines, etc., from the logarithmic sines, etc (log. sin x, etc.).

83. Application of the logarithmic function (52) or the equivalent transcendental function of y (53) are quite frequent in physical science. Thus the height H of a place where the height of the barometer is h is

$$H = 18415 (\log h - \log h') \tag{54}$$

meters over the place where the height of the barometer is h′. Compare El. Phys. 132. The resistance of the air, diminishing the amplitude of the oscilations of a pendulum, is also expressed by the logarithmic function. For students, experiments the pendulum ball should be of wood, and the length of the pendulum 2 to 4 meters. The elongation = y, the number of vibrations = x, are the two variables.

CURVES WITH FOCI.

84. The curve of probability, Fig. 1, C is also expressed by a rather complex transcendental equation, as shown in School Lab. 1872, p. 28. The experiments there described should carefully be made, and properly reduced.

85. In some problems of physics a curve is determined by certain special conditions in regard to the distance of each of its points from certain fixed points called geometrical foci and certain straight lines called directrices. We shall here only refer to the simplest cases of this kind.

First, if the distance of each point of a curve to a given fixed point is to be the same, the resulting curve is evidently the circle.

86. If each point M, Fig. 9, of the curve is to have a distance f from a focus directly proportional to its perpendicular distance p from a directrice AB, or if

$$\frac{f}{p} = e \tag{55}$$

then the resulting curve will be a conic section, namely, an ellipsis if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$. Construct the parabola, if the distance of F from AB is 2 cm. Also the ellipsis for $e = \frac{1}{2}$ and the hyperbola for $e = 2$ which correspond to the same NF and AB.

87. If the sum of the distances f and f′ each point M from two fixed foci F and F′ in the curve, Fig. 10, is to be constant, or if

$$f + f' = 2a \tag{56}$$

the resulting curve will be an ellipsis ME, of which F and F′ are the foci. If, however, the difference of the two distances is to be constant, or if

$$f - f' = 2a \tag{57}$$

then the resulting curve MH will be a hyperbola. Construct for $FF' = 8$ cm., and the sum $= 10$ cm. or the difference $= 6$ cm. Entire systems of such curves occur with circles in the wave motions of an elliptical vessel (Weber).

88. If the product of the distances f and f′ from the two foci F and F′ is to be constant for the same curve, or if

$$f.f' = a^2 \tag{58}$$

the resulting curve will be a the so-called Cassini's Ellipse, Fig. 11.

The form of this ellipse varies much with the distance $FF' = 2b$ between the two foci. For $a = b$ we obtain the loop l, the lemniscate; for smaller values of a we obtain 2 separate somewhat elliptical figures

m.m, one around each focus; if $a > b'$ we obtain the elliptical forms such as p and q.

It will readily be seen that the entire system of curves obtained for all possible values of a for the same two foci (b constant) will form the beautiful system of curves observed in binaxial crystals (El. Phys. 295).

89. If the point M, Fig. 12, are determined by

$$\cos f + \cos f' = c \tag{59}$$

the magnetic curves results for the magnetic poles F and F′; f and f′ being the angles which the lines MF and MF′ form with FF′. Compare your experiment, El. Phys. 310.

90. In many cases it is important to be able to represent the simultaneous variation of three variables, x, y, z. This can be done by taking two of these variables, for example x and y, to determine a point and marking the value of third z in number at the point. When this has been done, construct the lines of equal value Z. See most meteorological maps, where x and y are the latitude and longitude, f the pressure of the barometer (giving isobares), or the mean degree of temperature of the year (isothermes), etc., etc. See also El. Chem. Pl. 3, where x = specific gravity, y = hardness, and the curves are lines of equal chemical constitution. See especially the curves 11, 15, 181, 101.

VI. CONCLUSION.

The student who has faithfully worked his way through the few pages of this introduction to the method of inductive physics — a labor that will require several months — will be able to study any special branch of physics with profit, and will also be properly prepared for the study of any of the branches of mechanical technics. Modern technics is largely the product of modern

physics, although, as a matter of course, modern physics has also reaped many important advantages from the progress of technics.

It will now be seen that inductive physics does indeed make its results "most certain and exact," as KEPLER stated in 1619, in regard to his third law—which he found after seventeen years of labor. We would request each student of this little work to try and *discover* by the method given (see 28) this law of *Kepler*, from the facts which he had at his disposition. If the student in somewhat less than seventeen "hours" finds the law, he will know that the method here given is not only certain, but also a *time-saving* engine. We have by its means discovered the true and general relation between the pressure of saturated vapors and their temperature. The grand third law of Kepler, the corner-stone of theoretical astronomy, we beg the student to discover by the method given from the following *observed* values (according to *Mädler*) of the mean distances d of the planets from the sun, and their times t of revolution around the sun. The unit of d is the distance of the earth from the sun; t is given in days.

PLANET.	Distance, d	Time, t
Mercury	0.39	88.0
Venus	0.72	224.6
Earth	1.00	365.2
Mars	1.52	687.0
Jupiter	5.20	4332.6
Saturn	9.54	10759.2
Uranus	19.18	30686.7
Neptune	30.03	60117.4

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